SCHUBERT POLYNOMIALS, PIPE DREAMS AND BUMPLESS PIPEDREAMS

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1. Introduction

Schubert calculus is a type of enumerative geometry studying the intersections of linear subspaces, and was introduced by by Hermann Schubert in 1874 [Sch79]. One of the representative questions analyses intersections in projective 3-space. We can consider projective 3-space to be complex 3-space (with "points at infinity," such that any 2 hyperplanes intersect in a line. We are given four lines in general position, $\{\ell_1, \ell_2, \ell_3, \ell_4\}$. A set of lines is said to be in general position if no two lines are parallel, and no three lines intersect at one point.

We ask: how many lines intersect all four of our given lines?

The answer can be found to be two. Given that any pair of lines will intersect, arbitrarily create two pairs of lines: pair $P_A = \{\ell_A, \ell_B\}$ where ℓ_A and ℓ_B intersect at point I_A, and pair $P_B = \{\ell_C, \ell_D\}$ with intersection I_B. The line between I_A and I_B is one line intersecting all of $\{\ell_1, \dots, \ell_4\}$.

Recalling that these lines are in projective 3-space, therefore each of the pairs P_1 and P_2 span a hyperplane. These hyperplanes intersect in a line, giving our second line.

More generally, *Schubert varieties* are sets of k-dimensional subspaces, associated with partitions λ, that intersect another given subspace under specific conditions. If two varieties are associated with the same partition λ, they share the same *Schubert class*. Using Schubert classes, we can get the intersections of Schubert varieties with respect to various bases. We are interested in the coefficients associated with this procedure. We can use ***Schubert polynomials***, introduced by Lascoux and Schützenberger [LS82], to represent Schubert classes.

For these reasons, we are interested in finding Schubert polynomials. Calculating Schubert polynomials explicitly is difficult, but they can be defined recursively or combinatorially. In this paper, we will focus on how Schubert polynomials can be expressed using two combinatorial objects: ***pipe dreams*** [FK96; BJS93] and ***bumpless pipe dreams*** [LLS21]. For an example of a pipe dream and a bumpless pipe dream, see Figure 1. Schubert polynomials are indexed by permutations in w, which we can use to create (bumpless) pipe dreams.

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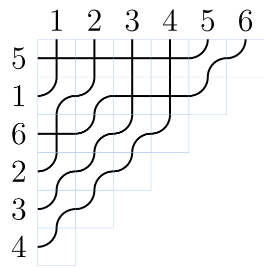
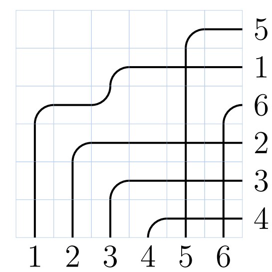


(A) a pipe dream for w representing $x_1^4 x_2^2 x_3$



(B) a bumpless pipe dream for w representing $x_1^4 x_2^2 x_3$

Figure 1. Representations of $w = 516234$
$\mathfrak{S}_w = x_1^4 x_2^3 + x_1^4 x_2^2 x_3 + x_1^4 x_2 x_3^2 + x_1^4 x_3^3$

Some permutations can have multiple pipe dreams. Each pipe dream is associated with a monomial in $\{x_1, x_2, \ldots, x_n\}$. Taking the sum over all pipe dreams for a permutation, we get the Schubert polynomial $\mathfrak{S}_w$.

Theorem 1.0.1. *The Schubert polynomial $\mathfrak{S}_w$ can be expressed as*

$$\mathfrak{S}_w = \sum_{\text{pipedreams for } w} \mathrm{wt}(\textit{pipe dream}),$$

where $\mathrm{wt}(\textit{pipe dream})$ *counts the number of intersections between pipes in a pipe dream.*

Given one pipe dream for a permutation w, we can generate all the other pipe dreams for w, and express the Schubert polynomial indexed by w. We prove that this is true in Theorem 3.2.1 using transformations on pipe dreams called chute and ladder moves [BB93].

Schubert polynomials can also be represented using ***bumpless pipe dreams*** [LLS21]. Bumpless pipe dreams correspond to the double Schubert polynomial generalisation of Schubert polynomials. Furthermore, there exists a bijection [GH23] between the set of bumpless pipe dreams for Schubert polynomials, and the pipe dreams.

In Section 2 of this paper, we begin with a few key definitions necessary for Schubert polynomials, as explored by Macdonald [Mac91] and Fomin and Stanley [FS91]. We discuss the recursive definition as given by Lascoux and Schützenberger [LS82]. In Section 3, we discuss a combinatorial description of Schubert polynomials via pipe dreams. In Section 4, we describe how Schubert polynomials can also be expressed using bumpless pipe dreams [LLS21]. We finish with brief details on the bijection between classical pipe dreams and bumpless pipe dreams [GH23].

2. Schubert Polynomials

Schubert polynomials were developed by Lascoux and Schuützenberger [LS82] and expanded upon by Macdonald [Mac91]. They are used to encode information about the intersections of linear subspaces, under various imposed restrictions.

Schubert polynomials $\mathfrak{S}_w$ are indexed by permutations $w \in S_n$ and are polynomials in $\{x_1, x_2, \ldots, x_{n-1}\}$.

Schubert polynomials have both a recursive and an explicit definition. We will cover the original recursive definition, and in the next section, introduce a combinatorial representation.

2.1. **Permutations.** The Schubert polynomial $\mathfrak{S}_w$ is indexed by permutation w in the symmetric group S_n. Given a permutation w, we can recursively define $\mathfrak{S}_w$ from $\mathfrak{S}_{w_0} = x_1^{n-1}x_2^{n-3}\dots x_2^2x_1$, which is indexed by the ***longest permutation*** in S_n: $w_0 = (n)(n-1)\cdots 21$.

We will define the longest permutation w_0 as the permutation with the most number of inversions. For further details on permutations, see [Mac91, Chapter 1].

Definition 2.1.1. An ***inversion*** is defined as occurring at index i if there exists $j >$ such that $w_i > w_j$. We count the total number of inversions over all possible pairs (i, j) as

$$\mathrm{inv}(w) = \sum_{i,j} \#\{\text{pairs } i < j \text{ and } w_i > w_j\}.$$

The longest permutation w_0 has the maximum possible $(i-1)$ inversions at every index i.

2.2. **Divided Difference Operators.** Given a permutation w, we can express the Schubert polynomial indexed by w using divided difference operators.

Definition 2.2.1 ([FS91])**.** Divided difference operators act on Schubert polynomials to get another Schubert polynomial. Given a Schubert polynomial $\mathfrak{S}_w$ indexed by w, let s_i be a simple transposition that swaps the variables x_i and x_{i+1} in $\mathfrak{S}_w$. The resulting Schubert polynomial is indexed by ws_i, where s_i swaps the indices i and $i+1$ in w. We define the i-th ***divided difference operator*** ∂_i that acts on Schubert polynomial $\mathfrak{S}_w$ to produce

$$\partial_i \mathfrak{S}_w = \frac{\mathfrak{S}_w - s_i\mathfrak{S}_w}{x_i - x_{i+1}}.$$

Example 2.2.2. For example, we can calculate $\mathfrak{S}_{231}$ from $\mathfrak{S}_{321}$. Permutation $231 = (321)s_1$, therefore we use divided difference operator ∂_1.

$$\begin{aligned}\mathfrak{S}_{321} &= x_1^2x_2,\\ \partial_1(\mathfrak{S}_{321}) &= \frac{x_1^2x_2 - x_2^2x_1}{x_1 - x_2} = \frac{(x_1x_2)(x_1-x_2)}{x_1-x_2},\\ &= x_1x_2 = \mathfrak{S}_{231}.\end{aligned}$$

2.3. **Schubert Polynomials.** We can recursively calculate all Schubert polynomials in n starting from the Schubert polynomial indexed by the longest permutation, $\mathfrak{S}_{w_0}$.

Definition 2.3.1. Given $w_0 \in S_n$ as $(n-1)(n-2)\dots 21$,

$$\mathfrak{S}_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}^1.$$

Definition 2.3.2 ([LS82])**.** For each $w \in S_n$, the Schubert polynomial $\mathfrak{S}_w$ is

$$\mathfrak{S}_w = \begin{cases}\mathfrak{S}_{w_0} & \text{if } w = (n)(n-1)\dots 21,\\ \partial_i \mathfrak{S}_{ws_i} & \text{if } \mathrm{inv}(w) < \mathrm{inv}(ws_i)\,.\end{cases}$$

We can apply divided difference operators to $\mathfrak{S}_{w_0}$, and subsequent Schubert polynomials, to obtain all Schubert polynomials in S_n.

Example 2.3.3. In Figure 2, we derive Schubert polynomials for S_3.

$$
\begin{tikzcd}[column sep=tiny]
 & \mathfrak{S}_{312} = x_1^2 x_2^1 x_3^0 = x_1^2 x_2 \arrow[dl, "\partial_1"'] \arrow[dr, "\partial_2"] & \\
\begin{array}{l} \mathfrak{S}_{231} = \partial_1 \mathfrak{S}_{321} \\ = \dfrac{x_1^2 x_2 - x_2^2 x_1}{x_1 - x_2} = x_1 x_2 \end{array} \arrow[d, "\partial_2"'] & & \begin{array}{l} \mathfrak{S}_{312} = \partial_2 \mathfrak{S}_{321} \\ = \dfrac{x_1^2 x_2 - x_1^2 x_3}{x_2 - x_3} = x_1^2 \end{array} \arrow[d, "\partial_1"] \\
\begin{array}{l} \mathfrak{S}_{213} = \partial_1 \mathfrak{S}_{231} \\ = \dfrac{x_1 x_2 - x_1 x_3}{x_2 - x_3} = x_1 \end{array} \arrow[dr, "\partial_1"'] & & \begin{array}{l} \mathfrak{S}_{132} = \partial_1 \mathfrak{S}_{312} \\ = \dfrac{x_1^2 - x_2^2}{x_1 - x_2} = x_1 + x_2 \end{array} \arrow[dl, "\partial_2"] \\
 & \mathfrak{S}_{123} = \partial_1 \mathfrak{S}_{213} = \partial_2 \mathfrak{S}_{132} = 1 &
\end{tikzcd}
$$

FIGURE 2. Schubert Polynomials for S_3

3. Pipe Dreams

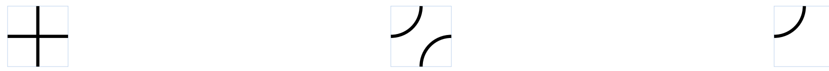
All Schubert polynomials can be combinatorially represented using ***pipe dreams***. Pipe dreams were first introduced as *rc-graphs* by Fomin and Kirillov [FK96], and Bergeron and Billey [BB93], and later coined "pipe dreams" by Knutson and Miller [KM05]. In this section, we discuss how to construct a pipe dream, and how to express its corresponding monomial. We will also discuss how to express a Schubert polynomial via transformations on a pipe dream.

3.1. Pipe Dreams.

Each pipe dream corresponds to a monomial in a Schubert polynomial $\mathfrak{S}_w$.

Definition 3.1.1 ([FK96; BB93])**.** Fix a permutation $w \in S_n$. A ***pipe dream*** for w is a grid of left-justified shape $(n, n-1, \ldots, 2, 1)$ in which "pipes" have been drawn as follows:

(1) Label the left-axis from top to bottom with $w_1, w_2, \ldots, w_n$.
(2) Label the top-axis from left to right, 1, 2, …, n.
(3) Draw pipes that move down and to the left that connect i on the top and $w_j = i$ on the left as in Figure 4 such that no two pipes cross more than once. Once all pipes are drawn, all tiles should be one of the three in Figure 3:
  (a) crossed (Figure 3a),
  (b) two elbow uncrossed (Figure 3b),
  (c) one elbow uncrossed (Figure 3c).
  The i-th tile in the i-th row can only be tiled with a one elbow uncrossing, otherwise there would be indices that are not connected by a pipe.



(A) Crossed (B) Two Elbow Uncrossed (C) One Elbow Uncrossed

FIGURE 3. Possible tiles in an pipe dream

Example 3.1.2. In Figure 4, we construct a pipe dream for $w = 312$ following Definition 3.1.1.

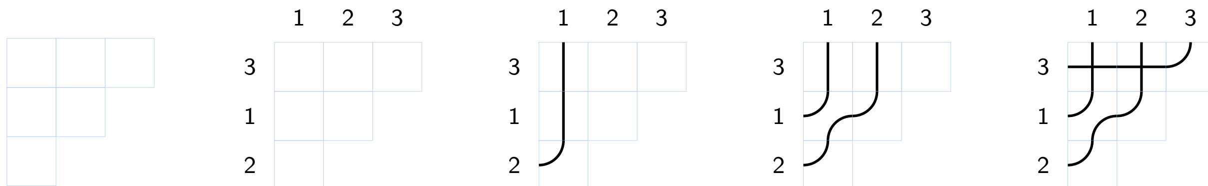


FIGURE 4. Constructing $\mathfrak{S}_{312} = x_1x_2$

Let cross($\wp$) be the set of crossed tiles in pipe dream $\wp$.

Definition 3.1.3. The ***weight*** wt($\wp$) of pipe dream $\wp$ is defined as a monomial in $\{x_1, x_2, \ldots, x_n\}$ where each crossed tile in row i contributes an x_i to the term

$$\mathrm{wt}(\wp) = \prod_{(j,i)\in \mathrm{cross}(\wp)} x_i.$$

We can express Schubert polynomial $\mathfrak{S}_w$ using the set of pipe dreams $\mathscr{P}(w)$.

Theorem 3.1.4. *Fix permutation $w \in S_n$. Let $\mathscr{P}(w)$ be the set of pipe dreams $\wp$ indexed by w. We can express the Schubert polynomial as a sum over its pipe dreams*

$$\mathfrak{S}_{\mathfrak{w}} = \sum_{\wp \in \mathscr{P}(w)} wt(\wp).$$

Example 3.1.5. In Figure 5, we present the pipe dreams for the Schubert polynomials in S_3.

3.2. **Chute & Ladder Moves.** We see that there exist two pipe dreams for $w = 132$, which show that $\mathfrak{S}_{w=132} = x_1 + x_2$.

Theorem 3.2.1 ([BB93])**.** *Given any w and one corresponding pipe dream, we can generate all other pipe dreams for w and express $\mathfrak{S}_w$, via chute moves and ladder moves.*

We will discuss a proof of Theorem 3.2.1 following [BB93]. Prior to this, we need to look at a few key definitions and lemmas.

Chute and ladder moves act on pipe dreams to move crossed tiles.

Definition 3.2.2 ([BB93])**.** For $m \geq 2$, consider a $(m+1) \times 2$ rectangular section of a pipe dream that spans $m+1$ columns $(j-m, \ldots, j)$ and 2 rows $(i, i+1)$. To perform a chute move, we require that

(1) the inner $m-1$ by 2 section, spanning columns $(j-m+1, \ldots, j-1)$ and rows $(i, i+1)$, is tiled with crossings.

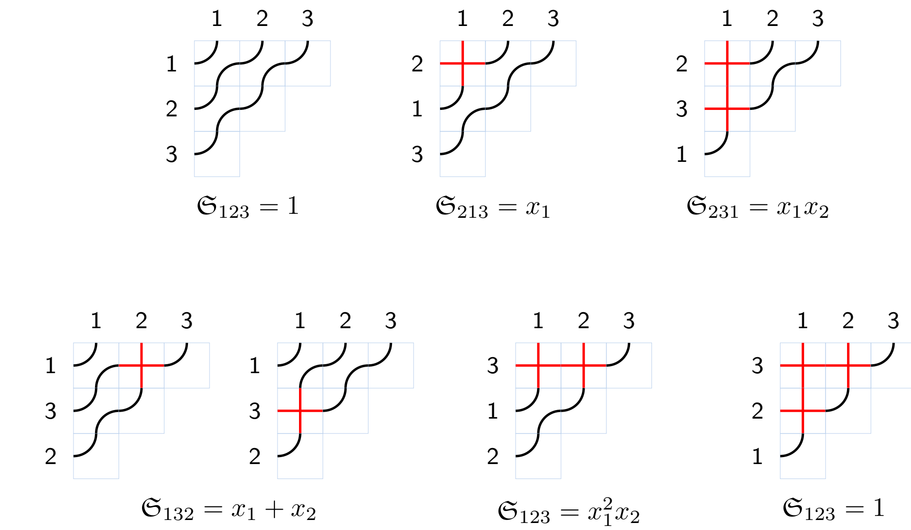


FIGURE 5. All pipe dreams in S_3

(2) locations $(j-m, i)$ and $(j-m, i+1)$ are tiled with a two elbows uncrossing. Location $(j, i+1)$ can be tiled with either a two elbow, or if $j = i+m$, a one elbow uncrossing.
(3) there is a crossing at (j, i).

A ***chute move*** $C_{i,j,m}$ is defined as a transformation that swaps the crossing at (j, i) with an uncrossing at $(j-m, i+1)$. An ***inverse chute move*** $C^{-}_{i,j,m}$ is similarly defined as the chute move, except pre-conditions include a crossed tile at $(j-m, i+1)$ and an uncrossed tile at (j, i). Inverse chute move $C^{-}_{i,j,m}$ swaps the crossing at $(j-m, i+1)$ with the uncrossing at (j, i). See Figure 6 for an example of a chute and inverse chute move.

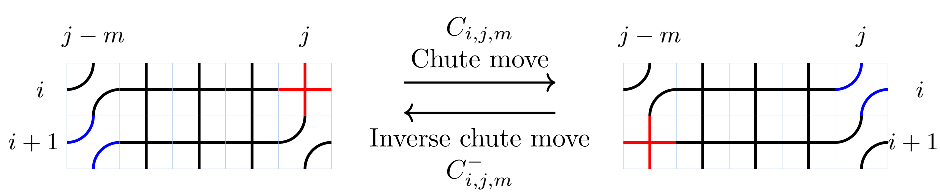


FIGURE 6. Valid chute and inverse chute move on a pipe dream [BB93]

Performing a chute move rotated 90° counterclockwise is known as a ladder move.

A ***ladder move*** $L_{i,j,m}$ is defined as swapping the crossing at (j, i) with an uncrossing at $(j-1, i+m)$. An ***inverse ladder move*** is similarly defined as a ladder move, except $L^{-}_{i,j,m}$ swaps a crossing at $(j-1, i+m)$ with an uncrossing at (j, i). See Figure 7 for an example of a ladder and inverse ladder move.

On a 2 by 2 square region, where $m = 2$, performing a chute move gives the same result as performing a ladder move. See Figure 8.

If you have a diagonal of two elbow tiles, with one crossed tile somewhere on the diagonal, then you can repeatedly perform this minimal chute/ladder to move the crossing up/down the diagonal.

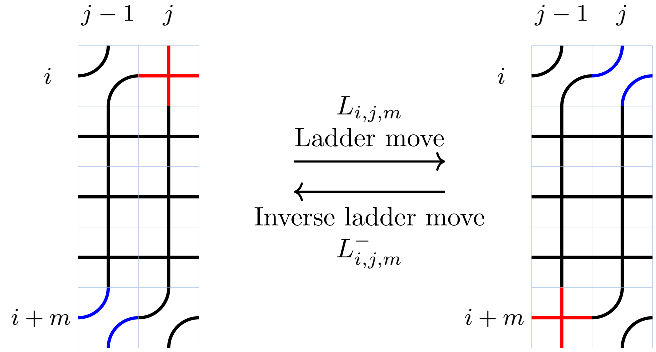


FIGURE 7. Valid ladder and inverse ladder move on a pipe dream, example from [BB93]

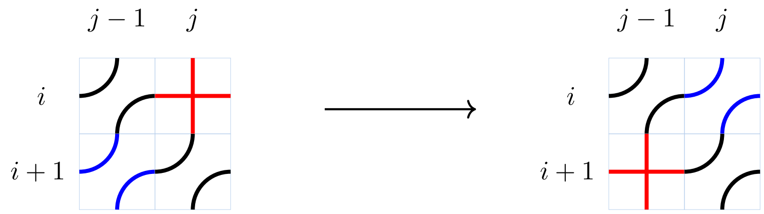


FIGURE 8. Chute move $C_{i,j,1}$ or ladder move $L_{i,j,1}$ on a 2×2 region

Lemma 3.2.3 ([BB93])**.** *Chute and ladder moves (and their inverses) preserve the permutation associated with an pipe dream.*

Example 3.2.4 ([BB93])**.** Consider $w = 1432$. In Figure 9, we generate $\mathscr{P}(w)$, starting with the pipe dream for w with all possible crossed tiles as far to the lower left as possible, $\mathcal{P}_{\text{left}}$. We can repeatedly perform inverse chute and ladder moves until we achieve the pipe dream with all intersections are as far to the upper right as possible, $\mathcal{P}_{\text{right}}$.

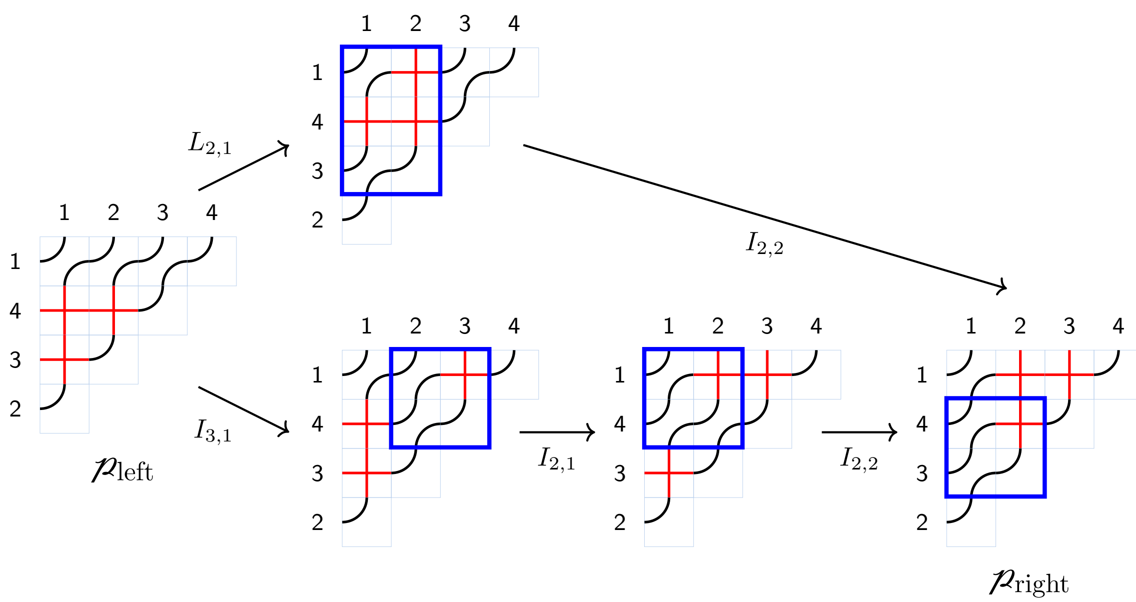


FIGURE 9. $\mathfrak{S}_{1432} = x_2^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1^2 x_3 + x_1^2 x_2$

We can now prove Theorem 3.2.1 using the following Lemmas 3.2.6 and 3.2.5

Lemma 3.2.5 ([BB93])**.** *A pipe dream $\mathcal{P}$ admits an inverse chute move if and only if there exists some uncrossed tile $(j, i) \notin$ cross$(\mathcal{P})$ such that $(j, i+1) \in$ cross$(\mathcal{P})$.*

Proof. Suppose there exists an uncrossed tile $(j, i) \notin \text{cross}(\mathcal{P})$ and crossed tile $(j, i+1) \in \text{cross}(\mathcal{P})$ as in Figure 10. We know that the pipes in $(j, i+1)$ have indices s and t, for $s < t$. If $t < s$, this implies that there is another crossing at some $(j - c_1, i + c_2)$, $c_1, c_2 \in \mathbb{Z}_+$, where pipe t is the horizontal pipe that crosses over pipe s. Then, pipe t would "above" pipe s until they cross again at (j, i). Two pipes are not allowed to cross twice by the definition of pipe dreams.

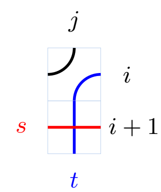


Figure 10. Uncrossing $(j, i) \notin \text{cross}(\mathcal{P})$
Pipes s and t cross at $(j, i+1)$

We then look along row $i+1$ for the largest $k_s > j$ such that $(i+1, k_s) \notin \text{cross}(\mathcal{P})$. We know that there are a finite number of crossed tiles in $\mathcal{P}$ and that the right-most tile in each row is always uncrossed. Therefore, k_s must exist.

We introduce a convention to refer to the separate elbows of a two elbow uncrossing. A *top elbow* is the elbow in the northwest corner of a tile, and a *bottom elbow* is the elbow in the southwest corner, as in Figure 11.

(A) Top Elbow (B) Bottom Elbow

Figure 11. Elbows in a Two Elbow Uncrossing

The tile $(k_t, i+1)$ directly underneath (k_t, i) has two possibilities:

The inequality $s < t$ implies that for some k_t, such that $j < k_t \leq k_s$, there must exist an two elbow uncrossing at $(k_t, i+1)$ where pipe t is the top elbow. If $k_s < k_t$, then pipes s and t would cross twice as in Figure 12, which is not allowed by the definition of pipe dreams. We can assume that the tiles in between columns j and k_t for both rows are crossings. If there was a two elbow uncrossing at some $t < k_r < k_t$, then replace k_t so that $k_t^{\text{new}} = k_r$.

The tile $(k_t, i+1)$ directly underneath (k_t, i) must be a two elbow (or one elbow) uncrossing, where pipe s is the top elbow, which gives $k_s = k_t$. Therefore, we can perform inverse chute move $C^-_{i,j,2}$ as in Figure 14. If there was a crossing, then this implies the configuration in Figure 13.

$\square$

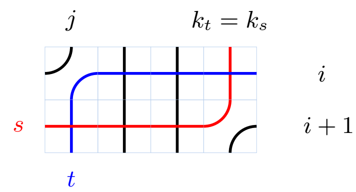


FIGURE 12. If $s < t$, then $k_s < k_t$ is not allowed

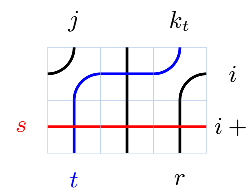


FIGURE 13. Pipe r replaces pipe t

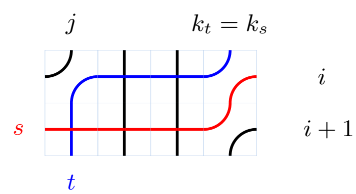


FIGURE 14. Pipe t forms a two elbow uncrossing with pipe s

The proof for Lemma 3.2.5 also applies for ladder moves, but instead of moving right along the columns, moves up the rows.

Lemmas 3.2.6 and 3.2.7 follow from Lemma 3.2.5.

Lemma 3.2.6 ([BB93])**.** *Pipe dream $\mathscr{p}_{right}$ does not admit an inverse move.*

Proof. In $\mathscr{p}_{\text{right}}$, each column j has some k_j crossings. By definition, these crossings occupy the top k_j rows of j. A crossing in row 1 blocks an inverse move from being performed on the crossing in row 2, and so on until row k. Because of these blocking tiles, no inverse moves can be performed on $\mathscr{p}_{\text{right}}$. □

Lemma 3.2.7. *For any given pipe dream $\mathscr{p} \neq \mathscr{p}_{right}$, there exists some finite sequence of chute moves $\mathscr{C}$ on $\mathscr{p}$ such that it terminates at $\mathscr{p}_{right}$.*

Symbolically, this gives the equality

$$\mathscr{C}(\mathscr{p}_{right}(w)) = \mathscr{P}(w).$$

Proof. We know that applying any sequence of chute moves to $\mathscr{p}_{\text{right}}$ gives a pipe dream in $\mathscr{P}(w)$, from Lemma 3.2.5. By definition, an inverse chute move move a crossing from the lower left of a pipe dream to the upper right direction. There are a finite number of locations that a crossing can be moved to, therefore, after a sequence of inverse chute moves, the crossings must accumulate in the upper right of the pipedream. This gives $C(\mathscr{p}_{\text{right}}(w)) = \mathscr{P}(w)$ and $\mathscr{C}(\mathscr{p}(w) = \mathscr{p}_{\text{right}}(w)$. □

Using Lemmas 3.2.6 and 3.2.7, we can conclude our original Theorem 3.2.1, restated here:

Theorem 3.2.1 ([BB93])**.** *Given any w and one corresponding pipe dream, we can generate all other pipe dreams for w and express $\mathfrak{S}_w$, via chute moves and ladder moves.*

4. Bumpless Pipe Dreams

The Schubert polynomials have another combinatorial representation, bumpless pipe dreams [LLS21]. Bumpless pipe dreams are so named for not allowing any "two elbows" tiling (see Figure 15). Unlike classical pipe dreams, bumpless pipe dreams can have empty tiles (see Figure 16a) in the interior of the graph. In addition, bumpless pipe dreams use the full n by n grid, as opposed to the $(n)(n-1)\dots 21$-shaped grid.

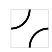


Figure 15. Forbidden in bumpless pipe dreams: Two elbows uncrossed

4.1. Bumpless Pipe Dreams.

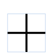
Definition 4.1.1 ([LLS21])**.** Fix $w \in S_n$. A ***bumpless pipe dream*** for w is a n by n grid, on which we draw pipes such that

1. Label the right axis from top to bottom with w.
2. Label the bottom axis from left to right with $1, 2, \dots, n$.
3. Draw pipes that move up and to the right that connect matching i on the bottom axis with $w_j = i$ on the right axis such that no two pipes cross twice. Once all pipes have been drawn, all tiles should be in the set of valid tiles (see Figure 16 for valid tiles).

(a) Blank

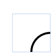
(b) Vertical

(c) Horizontal



(d) Crossed

(e) Top Elbow



(f) Bottom Elbow

Figure 16. Possible tiles in a bumpless pipe dream

Example 4.1.2. In Figure 17, we present a bumpless pipe dream for permutation 13254.

Definition 4.1.3. ([LLS21, Section 5.2] The ***Rothe*** pipe dream for w is the bumpless pipe dream where each pipe only bends once and has no top elbows. See Figure 18 for an example.

Let $\mathrm{blank}(\mathscr{b})$ be the set of locations of blank tiles in bumpless pipe dream $\mathscr{b}$ and $\mathscr{B}(w)$ represent the set of bumpless pipe dreams for w. We can define the weight of a bumpless pipe dream similarly to the weight of a pipe dream.

Definition 4.1.4. ([LLS21, Section 5.1]) The ***weight*** $\mathrm{wt}(\mathscr{b})$ of bumpless pipe dream $\mathscr{b}$ is defined as a monomial in $\{x_1, x_2, \dots, x_n\}$ where each blank tile in row i contributes an x_i to the term,

$$\mathrm{wt}(\mathscr{b}) = \prod_{(j,i)\in \mathrm{blank}(\mathscr{b})} x_i.$$

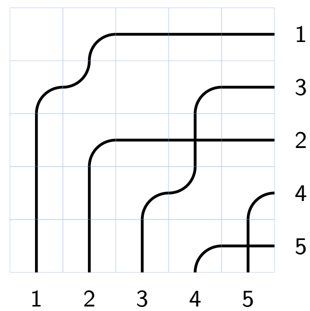


FIGURE 17. Bumpless pipe dream for $w = 13254$

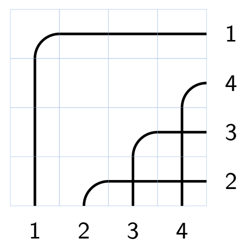


FIGURE 18. Rothe bumpless pipe dream for $w = 1432$

Theorem 4.1.5. *([LLS21, Theorem 5.2]) Fix $w \in S_n$. We can express Schubert polynomial $\mathfrak{S}_w$ using the set of bumpless pipe dreams for w:*

$$\mathfrak{S}_w = \sum_{\mathscr{b} \in \mathscr{B}(w)} wt(\mathscr{b}).$$

4.2. **Droop Moves.** Similar to classical pipe dreams, we can generate all other bumpless pipe dreams for w via transformations called droop moves that act on bottom elbows.

Definition 4.2.1. Consider a rectangular section of a bumpless pipe dream that spans columns $(j, j+m)$ and rows $(i, i+k)$. A droop move $D_{i,j,m,k}$ can be performed if

(1) there exists a bottom elbow at location (j, i),
(2) there exists an empty tile at $(j+m, i+k)$.

Droop move $D_{i,j,m,k}$ swaps the bottom elbow at (j, i) with the empty tile at $(j+m, i+k)$, as in Figure 19. An ***undroop move*** $D^{-}_{i,j,m,k}$ is defined similarly a droop move, except it swaps a bottom elbow at $(j+m, i+k)$ with an empty tile at (j, i).

Example 4.2.2. Let $w = 1432$. In Figure 20, we use droop moves to find all bumpless pipe dreams for 4132, starting with the Rothe bumpless pipe dream, $\mathscr{b}_{\text{Rothe}}$. Taking the sum over the set of bumpless pipe dreams, we can express Schubert polynomial $\mathfrak{S}_{1432} == x_2^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1^2x_2$.

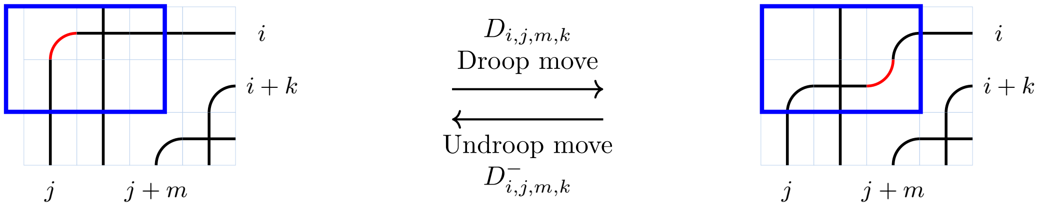


FIGURE 19. Droop move $D_{i,j,m,k}$

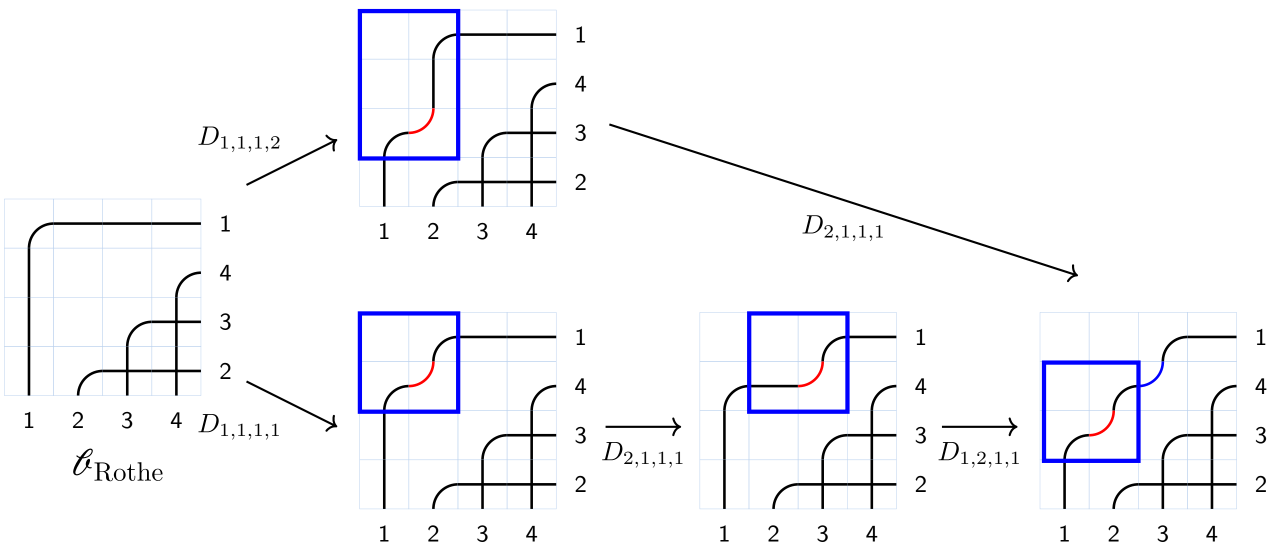


FIGURE 20. Schubert polynomial $\mathfrak{S}_{1432} = x_2^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1^2x_2$

Theorem 4.2.3 ([LLS21])**.** *Every bumpless pipedream for w can be obtained from the Rothe bumpless pipe dream $\mathcal{B}_{Rothe}(w)$ via some sequence of droop moves.*

For the proof of Theorem 4.2.3, see [LLS21, Proposition 5.3].

4.3. **Bijection between Bumpless Pipe Dreams and Classic Pipe Dreams.** Gao and Huang present a bijection between bumpless pipe dreams and pipe dreams [GH23].

Bumpless pipe dreams were initially conceptualised for a generalisation of the Schubert polynomials known as double Schubert polynomials. By using two variables, instead of just one as in Schubert polynomials, double Schubert polynomials encode further information about the linear subspaces being studied. For further treatment, see [Mac91] and [LLS21].

Theorem 4.3.1. *Given a bumpless pipe dream $\mathcal{B}(w)$, there exists a weight-preserving bijection between $\mathcal{B}(w)$ and a pipe dream $\mathcal{P}(w)$, such that $wt(\mathcal{B}(w)) = wt(\mathcal{P}(w))$.*

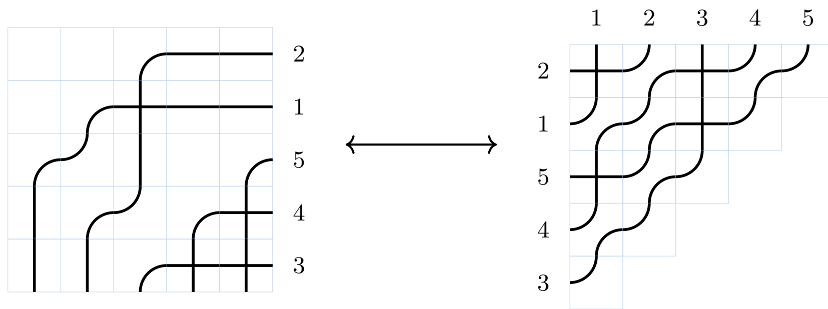


FIGURE 21. Mapping $\mathscr{P}_{21543} \longleftrightarrow \mathscr{p}_{21543}$ [GH23]

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